

Representing Euclidean Quantum Fields as Scaling Limit of Particle Systems

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We give a new representation of Euclidean quantum fields as scaling limits of systems of interacting, continuous, classical particles in the grand canonical ensemble.

KEY WORDS: Euclidean quantum field theory; Poisson random fields; interacting particle systems.

1. INTRODUCTION

The success of the Euclidean strategy in constructive quantum field theory (QFT) in the 1970s in many aspects was due to the formal analogy of Euclidean QFT with classical statistical mechanics, cf. ref. 1 and references therein. This analogy has been used in technically different approaches, as e.g., the lattice approximation,⁽¹⁾ the identification of the (massive) sine-Gordon model with the Coulomb (Yukawa) gas^(2,3) or, more recently, the polymer resp. random walk representation⁽⁴⁾ resp. ref. 5. In the present letter we introduce a new and conceptually different representation of Euclidean quantum fields by the means of systems of classical continuous particles in the grand canonical ensemble (GCE).

In ref. 6 it was realized that a Poissonization of the natural stochastic partial pseudo-differential equation associated to the Euclidean Bosonic free field⁽⁷⁾ leads to a new, in general not reflection positive, Euclidean QFT which however permits an analytic continuation (in the sense of quantum fields with indefinite metric⁽⁸⁾) from imaginary Euclidean time to

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real, relativistic time. Here we show that these Poisson Euclidean quantum fields can be identified with a noninteracting particle system in the grand canonical ensemble (Section 2).

Furthermore, using that the path-properties of Poisson Euclidean QFTs are more regular than in the standard Gaussian case, we introduce a class of ultra-violet (UV) finite, local interactions (in arbitrary dimension) for these models. On the level of classical particles moving in continuous space-time and described in the GCE, such field-theoretic interactions can be interpreted as a potential energy in the classical Hamiltonian of the n -particle system (Section 3).

Finally we identify a scaling limit under which the interacting Poisson Euclidean QFTs formally converge to the related perturbed Gaussian models. On the level of particle systems this corresponds to a scaling of activity and charge, while on the level of quantum fields there is a strong formal analogy with the block-spin formulation of the renormalization group, see e.g., ref. 5. We identify the change of path-properties under this limit and we give arguments to show, how in some simple situations this can be seen as a source of triviality, from which the necessity of a nontrivial renormalization arises (Section 4).

2. CONVOLUTED POISSON NOISE

Let $d \geq 2$ be the space-time dimension. The Euclidean (neutral, scalar, Bosonic) free field $\phi_0^g(x)$, $x \in \mathbb{R}^d$,⁽⁷⁾ of mass $m > 0$ in d dimensions³ can be obtained from the stochastic pseudo differential equation

$$(-\Delta + m^2)^{1/2} \phi_0^g(x) = \eta_0^g(x) \quad (1)$$

where η_0^g is a Gaussian white noise of intensity $\sigma > 0$, i.e., a centered Gaussian random field with covariance function $\sigma^2 \delta(x-y)$. Let $G(x)$ be the Green's function associated to $(-\Delta + m^2)^{1/2}$, then $\phi_0^g(x) = G * \eta_0^g(x)$ defines a pathwise solution of (1), i.e., a solution for any fixed random parameter. The convolution $G * \eta_0^g(x)$ is well-defined, since by Minlos' theorem⁽⁹⁾ $\eta_0^g(x)$ is a (random) tempered distribution and $(-\Delta + m^2)^{1/2}$ is continuously invertible on the space of tempered distributions.

Equation (1) can be modified by replacing the noise η_0^g by a Poisson noise as follows: For a bounded region $A \subseteq \mathbb{R}^d$ we define the Poisson noise random field on A as

$$\eta_{0,A}(x) = \sum_{j=1}^{N_A^z} S_{A,j} \delta(x - Y_{A,j}) \quad (2)$$

³ The subscript zero in this letter stands for noninteracting fields, not for time zero fields.

where N_A^z is a Poisson random variable with intensity $z |A|$, i.e., $P\{N_A^z = n\} = e^{-z|A|} (z|A|)^n / n!$. The parameter $z > 0$ is called the activity of the noise. $\{S_{A,j}\}_{j \in \mathbb{N}}$ is a family of independently identically distributed (i.i.d.) real-valued random variables with distribution given by a probability measure r on \mathbb{R} . For simplicity we assume that $\text{supp } r \subseteq [-c, c]$ for some $c > 0$. Finally, $\{Y_{A,j}\}_{j \in \mathbb{N}}$ is a family of \mathbb{R}^d -valued i.i.d. random variables distributed according to the uniform distribution on A .

We chose $A_n \subseteq \mathbb{R}^d$ a sequence of bounded, disjoint subsets of \mathbb{R}^d such that $\bigcup_{n=1}^\infty A_n = \mathbb{R}^d$. We then define

$$\eta_0(x) = \sum_{n=1}^\infty \eta_{0, A_n}(x) \tag{3}$$

where the noises $\eta_{0, A_n}(x)$ are independent from each other for different n . One can verify⁴ that the so-defined Poisson noise $\eta_0(x)$ does not depend on the choice of the sets A_n and that the restriction of $\eta_0(x)$ to a bounded set $A \subseteq \mathbb{R}^d$ has a representation (2). Furthermore, $\langle \eta_0, f \rangle = \int_{\mathbb{R}^d} \eta_0(x) f(x) dx$ for any Schwartz test function f exists with probability one, since $\eta_0(x)$ with probability one is polynomially bounded, as a consequence of the Borel–Cantelli lemma. $\langle \eta_0, f \rangle$ has infinitely divisible (probability) law and $\langle \eta_0, f \rangle$ and $\langle \eta_0, h \rangle$ are independent from each other whenever the test functions f and h have disjoint support. Also, $\eta_0(x)$ is invariant under Euclidean transformations (translations, reflections and rotations) in law.

These properties, which trivially also hold for the Gaussian white noise $\eta_0^g(x)$, show that from a mathematical point of view a legitimate modification of Eq. (1) is obtained if we replace the Gaussian noise $\eta_0^g(x)$ in (1) with $\eta_0(x)$. The solution $\phi_0(x) = G * \eta_0(x)$, constructed as above and called convoluted Poisson noise (CPN), is a Poissonian modification of the Euclidean free field.

This mathematical argument deserves a physical justification: As proven in ref. 6, the Euclidean quantum fields defined as CPN can be analytically continued to relativistic quantum fields with indefinite metric.⁽⁸⁾ Given that the covariance of CPN coincides with the covariance of the Euclidean free field,⁵ the asymptotic particle structure of these quantum fields⁽¹¹⁾ consists of free, relativistic particles of mass m and the scattering behavior is trivial, since mass-shell singularities of the higher order truncated Wightman functions are infra-particle like.⁽¹²⁾ In this sense CPN

⁴ One way to see this is to calculate the Fourier transform of $\langle \eta_0, f \rangle$, f a Schwartz test function, and show that it does not depend on A_n . The statement then follows from the uniqueness result of Minlos' theorem,⁽⁹⁾ cf. Section 3.2 of ref. 10 for the details.

⁵ With intensity $\sigma^2 = z^2 \int_{\mathbb{R}} s^2 dr(s)$. This should not be mixed up with the relations for the scaling limit in Section 4.

is still a scattering free field. These truncated Wightman functions can however be decomposed into direct integrals (over the mass-parameter) of Wightman functions with non-trivial scattering.^(6,11) Further motivation in favor of Poissonization will be given in Section 4.

To finish this section, we consider the path properties of the convoluted Poisson noise field $\phi_0(x)$: From the exponential decay $|G(x)| \sim e^{-m|x|}$ as $x \rightarrow \infty$ and the Borel–Cantelli Lemma one gets that with probability one the expression

$$\phi_0(x) = \sum_{n=1}^{\infty} \sum_{j=1}^{N_{A_n}^z} S_{j,n} G(x - Y_{A_n,j}) \quad (4)$$

is summable if x does not coincide with any of the $Y_{A_n,j}$. Only in the latter points the random field $\phi_0(x)$ has locally integrable singularities $\sim |x - Y_{A_n,j}|^{-d+1}$ as $G(x) \sim |x|^{-d+1}$ at 0. By this analysis we conclude that $\phi_0(x)$ with probability one is given by a locally integrable function. This is the basic difference with the Gaussian case, where the path properties of the Euclidean free field depend crucially on the dimension and for $d \geq 2$ cannot be represented by functions, cf. e.g., ref. 13.

3. LOCAL INTERACTIONS AND INTERPRETATION AS PARTICLE SYSTEMS

Let $v: \mathbb{R} \rightarrow \mathbb{R}$ be a function such that $|v'(\phi)| < C \quad \forall \phi \in \mathbb{R}$. Then v induces a (nonlinear) transform on the space of locally integrable functions⁶ via $\phi(x) \rightarrow v(\phi(x))$. Let now ν_0 be the probability measure on the space of locally integrable functions associated with the convoluted Poisson white noise $\phi_0(x)$.^(6,9) Then

$$d\nu_A(\phi) = \frac{\exp\{-\lambda \int_A v(\phi(x)) dx\}}{\int \exp\{-\lambda \int_A v(\phi(x)) dx\} d\nu_0(\phi)} d\nu_0(\phi) \quad (5)$$

defines a new measure ν_A on the space of locally integrable functions. Here, $A \subseteq \mathbb{R}^d$ is a bounded region and plays the role of an infra-red cut-off and $\lambda \geq 0$ is the coupling constant. As $v(\phi(x))$ is locally integrable, the expression (5)

⁶ Symbols as ϕ and η without subscripts and superscripts are used for sample paths (hence non-random objects) in the space of locally integrable functions and signed measures with finite support (in A), respectively, and are used as integration variables in (rigorous) integrals over such path-, respectively charge-, configurations. The random processes⁽⁹⁾ related to path- or configuration space measures (defining these integrals) are denoted by $\phi_0, \phi_{0,A}$ etc., resp. $\eta_0, \eta_{0,A}$, depending on the measure. Likewise, s_j, y_j, n etc. are non random coordinates parametrizing η while $S_{j,A}, Y_{j,A}, N_{A_n}^z$ are the related random objects parametrizing $\eta_{0,A}$

requires no further regularization, not even Wick ordering. Thus, for convoluted Poisson noise there exists a class of local potentials which is UV-finite independently of the dimension $d \geq 1$. This is in striking contrast to the Gaussian case. It should however be mentioned that the condition $|v'| < C$, which also implies that the denominator in (5) is finite, excludes polynomial potentials requiring regularization also in the Poisson case. However, trigonometric potentials, as the sine-Gordon potential, used in refs. 2 and 3 as perturbations of free fields, exist in our Poisson case independently of the dimension.

We now want to give a statistical mechanics interpretation to (5): $\eta_{0,\Lambda}(x)$ in (2) is interpreted as the random process describing N_Λ^z classical, indistinguishable and noninteracting particles with positions $Y_{\Lambda,j}$ in the “box” Λ . The particle with the position $Y_{\Lambda,j}$ carries the charge $S_{\Lambda,j}$. As N_Λ^z is Poisson distributed, the particle system is in the (configurational) grand canonical ensemble (GCE) with activity z . The Euclidean Poisson quantum field $\phi_{0,\Lambda}(x) = G * \eta_{0,\Lambda}(x)$ has a natural interpretation as the static field of the charge configuration $\eta_{0,\Lambda}(x)$, i.e., a unit charge in the point y gives rise to a static field $G(x - y)$ and the static field of a number of charges is then obtained by superposition.

Next we extend this statistical mechanics interpretation to the interacting case. Obviously, the interaction in (5) is given by the nonlinear energy density v of the static field. To formulate this on the level of the associated particle system, we find it more convenient to consider the particle system in the box $\eta_{0,\Lambda}(x)$ and the associated static field $\phi_{0,\Lambda}(x)$ instead of using the infra-red cut-off in (5). Identifying $\eta(x) = \sum_{j=1}^n s_j \delta(x - y_j)$ with the non-ordered n -tuple $\{(y_1, s_1), \dots, (y_n, s_n)\}$ we obtain as the interaction term in the n -particle Hamiltonian (for the normalization $v(0) = 0$)

$$\begin{aligned} U(\eta) &= \int_{\mathbb{R}^d} v(G * \eta(x)) \, dx \\ &= U(y_1, \dots, y_n; s_1, \dots, s_n) \\ &= \int_{\mathbb{R}^d} v\left(\sum_{j=1}^n s_j G(x - y_j)\right) \, dx \end{aligned} \tag{6}$$

where the integral exists due to the exponential decay of $G(x)$ as $|x| \rightarrow \infty$.

Let $\nu_{0,\Lambda}$ be the measure associated with the Euclidean quantum field $\phi_{0,\Lambda}(x)$ and let $\mu_{0,\Lambda}$ be the measure associated with the particle system in the box $\eta_{0,\Lambda}(x)$. We then define in analogy with (5), however, as we are in a finite volume, without additional infra-red cut-off

$$d\tilde{\nu}_\Lambda(\phi) = \frac{\exp\left\{-\lambda \int_{\mathbb{R}^d} v(\phi(x)) \, dx\right\}}{\int \exp\left\{-\lambda \int_{\mathbb{R}^d} v(\phi(x)) \, dx\right\} d\nu_{0,\Lambda}(\phi)} d\nu_{0,\Lambda}(\phi) \tag{7}$$

and likewise we define the Gibbs measure in the finite volume A as

$$d\tilde{\mu}_A(\eta) = \frac{\exp\{-\lambda U(\eta)\}}{\int \exp\{-\lambda U(\eta)\} d\mu_{0,A}(\eta)} d\mu_{0,A}(\eta) \quad (8)$$

where $\eta(x) = \sum_{j=1}^n s_j \delta(x - y_j)$ with $n \in \mathbb{N}_0$, $y_j \in A$ and s_j in the support of the measure r . Let $\tilde{\phi}_A(x)$ be the Euclidean quantum field associated with the measure $\tilde{\nu}_A$ and $\tilde{\eta}_A(x)$ be the interacting particle system associated with $\tilde{\mu}_A$. Then, using (6)–(8) one obtains by setting $\phi = G * \eta$ that $\tilde{\phi}_A(x)$ and $\tilde{\eta}_A(x)$ are related via the generalized Poisson equation (1), hence $\tilde{\phi}_A(x)$ is given by the static field of the interacting particle system $\tilde{\eta}_A(x)$.

One can now use techniques from the theory of Gibbs measures of classical continuous particle systems to study the infinite volume limit $A \uparrow \mathbb{R}^d$, cf. ref. 14 for the general strategy and ref. 10 for some first steps in that direction.

Interestingly, one can formulate a number of well-known systems of statistical mechanics as interacting Poisson quantum fields. Let $r = \delta_1$, i.e., all charges take the value $+1$. Then e.g., the choice $G = \chi_{B_R}$, the characteristic function of the ball centered at zero with radius R , and $v(\phi) = 0$ if $\phi < 2$, $v(\phi) = \infty$ if $\phi \geq 2$ gives a quantum field theoretic definition of the gas of hard spheres, cf. (6). Likewise, taking G an arbitrary smooth function of exponential decay with a singularity at 0 and $v(\phi) = 0$ if $\phi < C$, $C > 0$, and $v(\phi) = 1$ if $\phi \geq C$, respectively $v(\phi(x)) = \delta(\phi(x) - C) |\nabla\phi(x)|$, we obtain the field theoretic potentials defining the “trigger potential,” “isodensity contour potential” respectively, from stochastic geometry.⁽¹⁵⁾ Lastly, smearing out G with a mollifier, we can see that the quadratic potential $v(\phi) = \phi^2$ up to a chemical potential yields a particle potential with two point interactions.⁷ Removing the mollifier, the chemical potential diverges and has to be renormalized similarly as in classical electrostatics. Further details on the connections with classical statistical mechanics can be found in ref. 10.

4. THE GAUSSIAN SCALING LIMIT

In this section we consider the scaling limit $z \rightarrow \infty$ with field strength renormalization $\phi \rightarrow \phi/\sqrt{z}$. On the level of particle systems this can be seen as the scaling $z \rightarrow \infty$, i.e., the average number of particles per volume goes to infinity, and the charges of the particles scale with $1/\sqrt{z}$. We indicate this scaling with a superscript z .

⁷ In fact, a polynomial potential of degree n gives a potential which includes only j -point interactions for $j \leq n$. However, renormalization of such potentials for $n > 2$, when removing the mollifier, is more delicate.

If the particle system is noninteracting and the gas of particles is neutral in average, i.e., $\int_{\mathbb{R}} s \, dr(s) = 0$, then, by the central limit theorem, we get for a Schwartz test function f (note that N_A^z has expectation $z|A|$ and $S_{A,j} f(Y_{A,j})$ are independent, identically distributed random variables with expectation zero and variance $\int_{\mathbb{R}} s^2 \, dr(s) \int_A f^2(x) \, dx / |A|^2$),

$$\langle \eta_{0,A}^z, f \rangle = \frac{1}{\sqrt{z}} \sum_{j=1}^{N_A^z} S_{A,j} f(Y_{A,j}) \xrightarrow{\mathcal{L}} \langle \eta_{0,A}^g, f \rangle \quad (9)$$

where $\eta_{0,A}^g(x) = \chi_A(x) \eta_0^g(x)$, $\sigma^2 = \int_{\mathbb{R}} s^2 \, dr(s)$ and $\xrightarrow{\mathcal{L}}$ stands for convergence in probability law.

Likewise, using also (3) and (4), we get that $\langle \eta_0^z, f \rangle \xrightarrow{\mathcal{L}} \langle \eta_0^g, f \rangle$ and $\langle \phi_0^z, f \rangle \xrightarrow{\mathcal{L}} \langle \phi_0^g, f \rangle$. The latter scaling can equivalently be understood as a scale and mass transformation $x \rightarrow \alpha x$, $m \rightarrow m/\alpha$ in conjunction with a field strength renormalization $\phi \rightarrow \alpha^{(d-2)/2} \phi$, with $\alpha = z^{1/d} \rightarrow \infty$, cf. ref. 10. This clearly is analogous to the block-spin transformation in lattice Euclidean QFT.⁽⁵⁾ In the general sense that the renormalization group describes a process adding more and more micro-structures to a finite volume, we can thus consider the scaling $z \rightarrow \infty$ as an adequate implementation of the renormalization group in our framework.

Next, we want to consider the scaling for interacting models starting with UV-regularized models where G is replaced by $G_\epsilon = G * \chi_\epsilon$ with a mollifier $\chi_\epsilon \rightarrow \delta$ as $\epsilon \rightarrow +0$. We can construct a measure $\nu_A^{z,\epsilon}$ as in (5) where G is replaced by G_ϵ . As the theory now is UV-regularized, the related perturbed Gaussian measure $\nu_A^{g,\epsilon}$ exists. For simplicity we also assume that v is a bounded function. Let $\phi_A^{z,\epsilon}(x)$ and $\phi_A^{g,\epsilon}(x)$ be the Euclidean quantum fields (random fields) determined by the measures $\nu_A^{z,\epsilon}$ resp. $\nu_A^{g,\epsilon}$. We want to show that $\langle \phi_A^{z,\epsilon}, f \rangle \xrightarrow{\mathcal{L}} \langle \phi_A^{g,\epsilon}, f \rangle$ as $z \rightarrow \infty$, i.e., that the UV-regularized perturbed Poisson models converge to the related perturbed Gaussian ones. To see this, we consider the Fourier transform of $\langle \phi_A^{z,\epsilon}, f \rangle$, with $t \in \mathbb{R}$

$$\frac{\int \exp\{it\langle \phi, f \rangle - \lambda \int_A v(\phi(x)) \, dx\} \, d\nu_0^{z,\epsilon}(\phi)}{\int \exp\{-\lambda \int_A v(\phi(x)) \, dx\} \, d\nu_0^{g,\epsilon}(\phi)} \quad (10)$$

Expanding the numerator in increasing powers of λ we get as the n th coefficient

$$\frac{1}{n!} \int_{A^{xn}} \int \exp\{it\langle \phi, f \rangle\} v(\langle \phi, \chi_{\epsilon, y_1} \rangle) \cdots v(\langle \phi, \chi_{\epsilon, y_n} \rangle) \, d\nu_0^z(\phi) \, dy_1 \cdots dy_n \quad (11)$$

where $\chi_{\epsilon, y}(x) = \chi_\epsilon(x-y)$ and ν_0^z is the measure associated to $\phi_0^z(x)$ (where the latter random field has no UV-cut-off). As $\phi_0^z(x)$ converges in law to

$\phi_0^g(x)$ as $z \rightarrow \infty$, we can replace in (11) $dv_0^z(\phi)$ with $dv_0^g(\phi)$ in that limit. Furthermore, as the expansion in λ converges uniformly (as v is bounded) and the denominator has a corresponding expansion with $f = 0$, we can replace in (10) $dv_0^{z,\epsilon}(\phi)$ with $dv_0^{g,\epsilon}(\phi)$ as $z \rightarrow \infty$, which establishes our claim.

We now want to consider the same situation, but without UV-cut-off. Furthermore, we concentrate ourselves on trigonometric potentials⁽²⁾

$$v(\phi) = \int_{\mathbb{R}} \cos(\alpha\phi) d\rho(\alpha) \quad (12)$$

with ρ a finite signed measure on \mathbb{R} with $\rho\{0\} = 0$. One can then show that $\langle \phi_A^z, f \rangle \xrightarrow{\mathcal{L}} \langle \phi_0^g, f \rangle$, i.e., the scaling limit of the perturbed Poissonian model is Gaussian (and thus physically trivial).⁽¹⁰⁾ Here we want to give a qualitative argument to explain this statement: The average path $\phi(x)$ in the support of v_0^z with the increasing numbers of particles per volume of the associated particle system has more and more points where the path is behaving like $G(x)/\sqrt{z}$ for $x \rightarrow 0$. As G at 0 is not square integrable, these singularities have enough "volume" to make the average path more and more singular, as due to the process $z \rightarrow \infty$ more and more particles with charge $\sim \pm 1/\sqrt{z}$ are being added. Finally, all the area of A becomes covered with singularities and the average absolute value of

$$\int_A v(\phi(x)) dx = \int_{\mathbb{R}} \int_A \cos(\alpha\phi(x)) dx d\rho(\alpha) \quad (13)$$

becomes smaller and smaller as the oscillations in the inner integral integrate out to zero. In the limit $z \rightarrow \infty$ then the potential converges to zero (in law). This process elucidates the need of a renormalization procedure for the potential in the limit $z \rightarrow \infty$, i.e., in order to avoid triviality one has to consider energy densities v_z depending on the scaling parameter z .

The easiest case of such a renormalization is the coupling constant renormalization of the sine-Gordon model in $d = 2$ dimensions, see e.g., ref. 3. There, $v(\phi) = \cos(\alpha\phi)$ for α sufficiently small. To renormalize this potential (which otherwise would lead to a trivial scaling limit) one sets

$$v_z(\phi) = :\cos(\alpha\phi):^z = \frac{\cos(\alpha\phi)}{\int \cos(\alpha\phi(0)) dv_0^z(\phi)} \quad (14)$$

The $:^z$ -term in the middle is defined by the right hand side in the spirit of the Wick-ordering of the cosine function in the Gaussian case.^(2,3)

One can then show that in a certain expansion the characteristic functional (see (10)) of the process $\phi_A^z(x)$ with interaction given by v_z converges

in any order of that expansion to the related expression for the sine-Gordon model, cf. ref. 10 for further details.

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